

Extending the Nash Solution to Choice Problems with Reference Points

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Abstract

In 1985 Aumann axiomatized the Shapley NTU value by non-emptiness, efficiency, unanimity, scale covariance, conditional additivity, and independence of irrelevant alternatives. We show that, when replacing unanimity by “unanimity for the grand coalition” and translation covariance, these axioms characterize the Nash solution on the class of n -person choice problems with reference points. A classical bargaining problem consists of a convex feasible set that contains the disagreement point here called reference point. The feasible set of a choice problem does not necessarily contain the reference point and may not be convex. However, we assume that it satisfies some standard properties. Our result is robust so that the characterization is still valid for many subclasses of choice problems, among those is the class of classical bargaining problems. Moreover, we show that each of the employed axioms – including independence of irrelevant alternatives – may be logically independent of the remaining axioms.

Keywords: Bargaining problem, Nash set, Shapley NTU value

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1 Introduction

A *bargaining problem* on a set N of n agents consists of a pair (S, d) where $S \subseteq \mathbb{R}^N$, the feasible set, is a nonempty comprehensive set of utility profiles and $d \in S$ is the *disagreement point*. Nash (1950) characterized the *Nash solution* on the class of bargaining problems with convex feasible sets by *Pareto efficiency* (EFF), *equal treatment of equals*, *scale covariance* (SCOV), and *independence of irrelevant alternatives* (IIA). Bargaining problems may be regarded as particular *cooperative non transferable utility* (NTU) *games*, and, hence, the Shapley NTU value (Shapley 1969) is a generalization of the Nash solution. Aumann (1985) characterized the Shapley NTU

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value (on the class of NTU games with closed, non-leveled, and smooth feasible sets) by *non-emptiness* (NE), EFF, *conditional additivity* (CADD), *unanimity* (UNA), SCOV, and¹ IIA. For $n = 2$, an NTU game is either a bargaining problem or a *rationing problem*, i.e., a pair (S, c) where S is an aforementioned feasible set and $c \in \mathbb{R}^N$, the profile of *reference utilities*, does **not** belong to S . The current paper extends the rationing solution of Mariotti and Villar (2005) to non-convex rationing problems (see Theorem 3.1), and Proposition 5.1) also extends Proposition 4 of Herrero and Villar (2010).

Several authors investigated generalizations of the Nash solution for bargaining problems with non-convex feasible sets (Herrero 1989, Conley and Wilkie 1996, Zhou 1996, Mariotti 1998, Peters and Vermeulen 2010). We consider both, bargaining problems (i.e., surplus sharing problems) and rationing problems (i.e., bankruptcy problems), and do not assume in general that a feasible set is convex. Thus, this paper generalizes the definition of the Nash solution and extends its characterization so that it may be applied both to bargaining and to rationing problems whose feasible sets are not necessarily convex. Hence, the class of considered problems is much larger than the standard one. Moreover, our results turn out to be robust in the sense that they are applicable to several remarkable subsets of choice problems with reference points, e.g., to classical bargaining problems with convex feasible sets.

Rubinstein and Zhou (1999) introduce “choice problems with reference points”, i.e., pairs (P, r) where P is a set of feasible alternatives in some set X and r is a “reference point” in X . They do not assume that in their model the agents are endowed with any preference relations over the set X , i.e., they do not intend to model conflicting interests in a bargaining problem. In contrast to these authors, we specify these conflicting interests explicitly by considering the elements of P and r itself as utility profiles. Despite of these differences in the interpretations, bargaining and rationing problems have the same structure: They consist of a feasible set (of utility profiles) and a reference utility profile. Thus, we keep the name and call a pair (P, r) where $P \subseteq \mathbb{R}^N$ is a feasible set and $r \in \mathbb{R}^N$ a *choice problem* (on N) with *reference point* (r) as well. The Nash solution of a choice problem with reference point (P, r) is the Shapley NTU value of the associated NTU game if P is non-leveled, smooth, and convex: A Pareto efficient element x of P is a Shapley NTU value if and only if, up to some natural scaling, it is the Shapley value of the *transferable utility* TU game that arises when replacing P by the half-space determined by the supporting hyperplane through x . Hence, the concept of Nash solution may be generalized to a choice problem (P, r) with a non-convex (but still non-leveled and smooth) feasible set by replacing “supporting hyperplane” with “tangent hyperplane” where necessary. We show that suitable versions of the axioms used by Aumann may be used to characterize the aforementioned generalization of the Nash solution to several classes of choice problems with reference points. The axiom UNA requires for NTU games that the solution equally divides the available one unit to the players of S in the *unanimity game* of any subcoalition S of N . Thus,

¹His additional axiom of “Closure Invariance” is not relevant for NTU games with closed feasible sets.

the unique unanimity game that may be regarded as a choice problem with reference point (in fact a bargaining problem) is the unanimity game of the grand coalition N . Hence, UNA and SCOV on choice problems do not longer imply *translation covariance* (TCOV) as in the case of NTU games. Thus, in our results TCOV is additionally employed.

The paper is organized as follows. In Section 2 the necessary notation is provided, the definitions are presented, and straightforward properties of the Nash solution are deduced or recalled. It is also shown that the Nash solution of a *uniformly p-smooth* (in the sense of Maschler and Owen (1992)) choice problem with reference point always exists.

Section 3 is devoted to the axiomatization of the Nash solution on uniformly p-smooth choice problems with reference points. It is shown that Aumann's (1985) characterization of the NTU Shapley value may be resembled: The Nash solution is characterized by NE, EFF, CADD, UNA, SCOV, TCOV, and IIA (see Theorem 3.1). In Section 4 it is shown by means of examples that each of the employed axioms is logically independent of the remaining six axioms.

Section 5 investigates the “robustness” of Theorem 3.1. It turns out that the characterization still holds on many interesting classes of choice problems with reference points. Indeed the result holds for many classes of choice problems with convex feasible sets; e.g., for the class of classical bargaining problems with non-leveled smooth convex feasible sets, the Nash solution is characterized by the first six axioms, i.e., IIA is not needed. Also it turns out that, if IIA is slightly modified, the set of *uniformly p-smooth* choice problems may be replaced by the set of *p-smooth* choice problems with a nonempty Nash solution. Finally, this section contains some remarks about the logical independence of the axioms that are employed in the aforementioned modifications of Theorem 3.1.

The appendix (Section A) is devoted to show some technical results that are used in the preceding sections. Namely, it is shown that for any point x of its boundary, any (uniformly) p-smooth feasible set contains a **convex** (uniformly) p-smooth feasible set that contains x .

2 Preliminaries

For a finite set N we denote by \mathbb{R}^N the set of all real functions on N . So \mathbb{R}^N is the $|N|$ -dimensional Euclidean space. (Here and in the sequel, if D is a finite set, then $|D|$ denotes the cardinality of D .) For $x, y \in \mathbb{R}^N$ let $x \cdot y$ denote the scalar product, and we write $x \geq y$ if $x_i \geq y_i$ for all $i \in N$. Moreover, we write $x > y$ if $x \geq y$ and $x \neq y$ and we write $x \gg y$ if $x_i > y_i$ for all $i \in N$. We denote $\mathbb{R}_+^N = \{x \in \mathbb{R}^N \mid x \geq 0\}$ and $\mathbb{R}_{++}^N = \{x \in \mathbb{R}^N \mid x \gg 0\}$. For every $S \subseteq N$, the *indicator function* on S is denoted by $\chi^S \in \mathbb{R}^N$, i.e.,

$$\chi_j^S = \begin{cases} 1, & \text{if } j \in S, \\ 0, & \text{if } j \in N \setminus S, \end{cases}$$

and $x(S) = x \cdot \chi^S = \sum_{i \in S} x_i$ for every $x \in \mathbb{R}^N$.

Let N be a finite nonempty set. A *choice problem* of N is a pair (P, r) such that $r \in \mathbb{R}^N$ and $P \subseteq \mathbb{R}^N$ is *feasible* (for N), i.e.,

$$P \neq \emptyset, \mathbb{R}^N; \quad (2.1)$$

$$P \text{ is closed}; \quad (2.2)$$

$$P \text{ is comprehensive: } x \in P, y \in \mathbb{R}^N, y \leq x \Rightarrow y \in P. \quad (2.3)$$

We recall the definition of “uniform positive smoothness” (Maschler and Owen 1989).

Definition 2.1 Let $P \subseteq \mathbb{R}^N$ be feasible. Denote by ∂P the boundary of P . The set P is **smooth** if

- (1) at every $x \in \partial P$ there exists a unique tangent hyperplane $H^x = H^{x,P}$ to P ;
- (2) the mapping $\partial P \rightarrow \mathbb{R}_+^N, x \mapsto \lambda^x = \lambda^{x,P}$, is continuous, where $\lambda^x \in \mathbb{R}^N$ is the unique normal vector determined by the requirements $\lambda^x(N) = 1$ and $H^x = \{y \in \mathbb{R}^N \mid \lambda^x \cdot y = \lambda^x \cdot x\}$.

Moreover, a smooth set P is **positively smooth** (**p-smooth**) if $\lambda^{x,P} \gg 0$ for all $x \in \partial P$, and it is **uniformly p-smooth** if there exists $\delta > 0$ such that $\lambda^{x,P} \gg \delta \chi^N$ for all $x \in \partial P$.

Thus, a choice problem (P, r) is *(uniformly p-)smooth* if P is (uniformly p-)smooth. Note that p-smoothness of P implies *non-levelness* of P , an assumption that is widely used in the corresponding literature.

Let N be a finite nonempty set. We denote by Γ_N the set of all p-smooth choice problems (P, r) such that $P \subseteq \mathbb{R}^N$. A *solution* on $\Gamma \subseteq \Gamma_N$ is a mapping σ that assigns to each $(P, r) \in \Gamma$ a set $\sigma(P, r) \subseteq P$. Some well-known properties of a solution σ on Γ are as follows. The solution σ satisfies

- (1) *non-emptiness* (NE) if $\sigma(P, r) \neq \emptyset \forall (P, r) \in \Gamma$;
- (2) *efficiency* (EFF) if $\sigma(P, r) \subseteq \partial P \forall (P, r) \in \Gamma$;
- (3) *conditional additivity* (CADD) if $(P^i, r^i) \in \Gamma, i \in \{1, 2, 3\}, P^3 = P^1 + P^2$, and² $r^3 = r^1 + r^2$ imply that

$$\sigma(P^3, r^3) \supseteq (\sigma(P^1, r^1) + \sigma(P^2, r^2)) \cap \partial P^3;$$

- (4) *unanimity* (UNA) if $(U^N, 0) \in \Gamma$ implies $\sigma(U^N, 0) = \left\{ \frac{1}{|N|} \chi^N \right\}$, where

$$U^N = \{x \in \mathbb{R}^N \mid x(N) \leq 1\};$$

- (5) *scale covariance* (SCOV) if for all $(P, r) \in \Gamma$ and for all $\lambda \in \mathbb{R}_{++}^N$ such that³ $(\lambda * P, \lambda * r) \in \Gamma$,

²Whenever applied to sets, the “+” denotes the “Minkowski sum”.

³We use the notation $\lambda * x = (\lambda_i x_i)_{i \in N}$ for all $\lambda, x \in \mathbb{R}^N$.

$$\sigma(\lambda * P, \lambda * r) = \lambda * \sigma(P, r);$$

- (6) *translation covariance* (TCOV) if for all $(P, r) \in \Gamma$ and $y \in \mathbb{R}^N$ such that $(P + \{y\}, r + y) \in \Gamma$, $\sigma(P + \{y\}, r + y) = \sigma(P, r) + \{y\}$;
- (7) *independence of irrelevant alternatives* (IIA) if $(P, r), (P', r) \in \Gamma$, $P \subseteq P'$ imply that $\sigma(P, r) \supseteq \sigma(P', r) \cap P$.
- (8) *independence of irrelevant expansions* (IIE) if $(P, r), (P', r) \in \Gamma$, $P \subseteq P'$ imply that $\sigma(P', r) \supseteq \sigma(P, r) \cap \partial P'$.

Definition 2.2 Let $(P, r) \in \Gamma_N$. The **(generalized) Nash solution** of (P, r) , denoted by $\phi(P, r)$, is defined by

$$\phi(P, r) = \{x \in \partial P \mid (x_i - r_i)\lambda_i^{x,P} = (x_j - r_j)\lambda_j^{x,P} \ \forall i, j \in N\}. \quad (2.4)$$

Let $(P, r) \in \Gamma_N$. If $r \notin P$, i.e., r is not feasible, then (P, r) is a *rationing problem* (that is frequently called “bankruptcy problem” in the literature) and

$$\phi(P, r) = \left\{ x \in \partial P \mid x \leq r, \prod_{i \in N} (r_i - x_i) \geq \prod_{i \in N} (r_i - y_i) \ \forall y \in H^{x,P} \cap \{z \in \mathbb{R}^N \mid z \leq r\} \right\}. \quad (2.5)$$

If $r \in P$, i.e., r is feasible, then (P, r) is a *bargaining problem* (that is also called “surplus sharing problem” in the literature) and

$$\phi(P, r) = \left\{ x \in \partial P \mid x \geq r, \prod_{i \in N} (x_i - r_i) \geq \prod_{i \in N} (y_i - r_i) \ \forall y \in H^{x,P} \cap \{z \in \mathbb{R}^N \mid z \geq r\} \right\}. \quad (2.6)$$

Note that the generalized Nash solution selects the set of stationary points of the *Nash product* $\prod_{i \in N} (x_i - r_i)$ over the set of Pareto efficient points of P .

Remark 2.3 Let $(P, r) \in \Gamma_N$. For $x \in \partial P$ define the TU game $(N, v) = (N, v_r^{x,P})$ by $v(N) = \lambda^{x,P} \cdot x$ and $v(S) = \lambda_S^{x,P} \cdot r_S$ for all $S \subsetneq N$. Note that $x \in \phi(P, r)$ if and only if $\lambda^{x,P} * x$ is the Shapley value of (N, v) (Shapley 1969).

In general, it is well-known that a generalized Nash solution may not exist even in the case that $N = \{1, 2\}$. Indeed, if (P, r) is defined by $r = 0$ and $P = \{x \in \mathbb{R}^N \mid x_1 < 0 \text{ and } x_2 \leq -\frac{1}{x_1^2}\}$, then $\phi(P, r) = \emptyset$. Thus, we denote

$$\begin{aligned} \Gamma_N^\phi &= \{(P, r) \in \Gamma_N \mid \phi(P, r) \neq \emptyset\} \text{ and} \\ \Gamma_N^{\text{uni}} &= \{(P, r) \in \Gamma_N \mid P \text{ is uniformly p-smooth}\}. \end{aligned}$$

We now show that $\Gamma_N^{\text{uni}} \subseteq \Gamma_N^\phi$.

Proposition 2.4 *The generalized Nash solution of a uniformly p-smooth choice problem exists.*

Proof: Let $(P, r) \in \Gamma_N^{\text{uni}}$. The following three claims may be used to finish the proof:

- (1) If $r \in \partial P$, then $\phi(P, r) = \{r\}$.
- (2) If $r \notin P$ and $\max_{x \in P, x \leq r} \prod_{i \in N} (r_i - x_i)$ exists, then

$$\emptyset \neq \arg \max \left\{ \prod_{i \in N} (r_i - x_i) \mid x \in \partial P, x \leq r \right\} \subseteq \phi(P, r) \subseteq \{x \in \partial P \mid x \ll r\}.$$

- (3) If r is in the interior of P and $\max_{x \in P, x \geq r} \prod_{i \in N} (x_i - r_i)$ exists, then

$$\emptyset \neq \arg \max \left\{ \prod_{i \in N} (x_i - r_i) \mid x \in P, x \geq r \right\} \subseteq \phi(P, r) \subseteq \{x \in \partial P \mid x \gg r\}.$$

Indeed, by Claim (1) we may assume that $r \notin \partial P$. If $r \notin P$, then $\{x \in \partial P \mid x \leq r\}$ is nonempty and compact by uniform p-smoothness so that $\max_{x \in P, x \leq r} \prod_{i \in N} (r_i - x_i)$ exists and Claim (2) finishes the proof. In the remaining case we may employ Claim (3).

We now show the three claims. The first claim is a straightforward consequence of the definition. The second inclusion in both of the remaining claims follows immediately from the definition of ϕ . The first inclusions may be shown as follows. If $r \notin P$, then let $y \in \arg \max_{x \in P, x \leq r} \prod_{i \in N} (r_i - x_i)$. The hyperplane $\{z \in \mathbb{R}^M \mid \lambda \cdot z = \lambda \cdot y\}$ is a tangent to the hyperbola $\{z \in \mathbb{R}^N \mid r \gg z, \prod_{i \in N} (r_i - z_i) = t\}$ so that $y \in \phi(P, r)$ by Remark 2.3. If r is in the interior of P , then the proof is similar. **q.e.d.**

Remark 2.5 Without smoothness the Nash solution violates CADD (Aumann 1985, Section 9, in particular Figure 1). If P and Q are smooth feasible sets, $x \in P, y \in Q$, and $x + y \in \partial(P + Q)$, then $x \in \partial P, y \in \partial Q$, and the tangent hyperplanes at x to P and at y to Q are parallel, i.e., $\lambda^{x,P} = \lambda^{y,Q}$.

From Definition 2.2 and the subsequent paragraphs it may easily be deduced that the generalized Nash solution satisfies the eight aforementioned properties on suitable subsets of Γ_N which is summarized in the following corollary.

Corollary 2.6 *The Nash solution on any $\Gamma \subseteq \Gamma_N^\phi$ satisfies NE, EFF, CADD, UNA, SCOV, TCOV, IIA, and IIE.*

Remark 2.7 Let $(P, r) \in \Gamma_N$.

- (1) Classically (Nash 1950) it was assumed that $r \in P$ and that P is a convex set. In this case, $\phi(P, r)$ is a singleton, namely the vector y that maximizes the Nash product $\prod_{i \in N} (x_i - r_i)$ subject to $x \in P$ that satisfy $x \geq r$.
- (2) Similarly, if $\mathbb{R}^N \setminus P$ is convex and $r \notin P$, then $\phi(P, x)$ consists of the unique vector that maximizes $\prod_{i \in N} (r_i - x_i)$ subject to $x \leq r$ and $x \in \partial P$ (see Eq. (2.6)).

3 Axiomatization on uniformly p-smooth choice problems

The main result of this section is the following theorem.

Theorem 3.1 *The Nash solution on Γ_N^{uni} is the unique solution that satisfies NE, EFF, CADD, UNA, SCOV, TCOV, and IIA.*

Proof: By Corollary 2.6 we only have to show the uniqueness part. Let σ be a solution on Γ_N^{uni} that satisfies the desired axioms. Let $(P, r) \in \Gamma_N^{\text{uni}}$. It remains to show that $\sigma(P, r) = \phi(P, r)$.

We consider first the following special cases.

- (1) $P = \{x \in \mathbb{R}^N \mid x(N) \leq 0\}, r = 0$: Let $x \in \sigma(P, r)$. By EFF, $x(N) = 0$. By CADD, $x + \frac{1}{|N|}\chi^N \in \sigma(U^N, 0)$. By UNA, $x = 0$. By NE, $\sigma(P, r) = \{0\} = \phi(P, r)$.
- (2) There exists $\lambda \in \mathbb{R}_{++}^N$ such that $P = \{x \in \mathbb{R}^N \mid \lambda \cdot x \leq \lambda \cdot r\}$. In this case $\sigma(P, r) = \phi(P, r) = \{r\}$ by Case 1, SCOV, and TCOV.
- (3) There exist $\lambda \in \mathbb{R}_{++}^N$ and $c > \lambda \cdot r$ such that $P = \{x \in \mathbb{R}^N \mid \lambda \cdot x \leq c\}$. Let

$$P' = \{x \in \mathbb{R}^N \mid \lambda \cdot x \leq c - \lambda \cdot r\}.$$

By SCOV and UNA, $\sigma(P', 0) = \phi(P', 0) = \left\{ \left(\frac{c - \lambda \cdot r}{|N| \lambda_i} \right)_{i \in N} \right\}$. By TCOV, $\sigma(P, r) = \sigma(P', 0) + \{r\} = \phi(P, r)$.

- (4) There exist $\lambda \in \mathbb{R}_{++}^N$ and $c < \lambda \cdot r$ such that $P = \{x \in \mathbb{R}^N \mid \lambda \cdot x \leq c\}$. Let

$$P' = \{x \in \mathbb{R}^N \mid \lambda \cdot x \leq -c\} \text{ and } P'' = \{x \in \mathbb{R}^N \mid \lambda \cdot x \leq 0\}.$$

By EFF and CAD, $\sigma(P', -r) + \sigma(P, r) \subseteq \sigma(P'', 0)$. By Case 1 and SCOV, $\sigma(P'', 0) = \{0\} = \phi(P'', 0)$. By Case 3 and NE, $\sigma(P, r) = -\sigma(P', -r) = \phi(P, r)$.

- (5) P is convex. Let $y \in \sigma(P, r)$. By EFF, $y \in \partial P$. Let $\lambda = \lambda^{y, P}$. Then $\lambda \cdot y \geq \lambda \cdot x$ for all $x \in P$ by convexity. Let

$$P' = \{x \in \mathbb{R}^N \mid \lambda \cdot x \leq \lambda \cdot y\} \text{ and } P'' = \{x \in \mathbb{R}^N \mid \lambda \cdot x \leq 0\}.$$

By Cases (2) – (4), $\sigma(P'', 0) = \{0\} = \phi(P'', 0)$ and $\sigma(P', r) = \phi(P', r)$ is a singleton. By CADD, $\{y\} + \sigma(P'', 0) \subseteq \sigma(P', r)$ so that $y \in \phi(P, r)$ by IIA of ϕ . The opposite inclusion is shown similarly by interchanging the roles of σ and ϕ .

- (6) $\mathbb{R}^N \setminus P$ is convex. Let $y \in \sigma(P, r)$. By EFF, $y \in \partial P$. Let $\lambda = \lambda^{y, P}$ and $P' = \{x \in \mathbb{R}^N \mid \lambda \cdot x \leq \lambda \cdot y\}$. By IIA and Cases (2) – (4), $y \in \sigma(P', r) = \phi(P', r)$. By IIE of ϕ (see Corollary 2.6), $y \in \phi(P, r)$.

In order to show that $\phi(P, r) \subseteq \sigma(P, r)$, let $z \in \phi(P, r)$. If r is not in the interior of P , then, by Remark 2.7 (2), $|\phi(P, r)| = 1$ so $z \in \sigma(P, r)$ by NE.

Hence, we may assume that r is in the interior of P . It is sufficient to find $(P', r), (P'', 0) \in \Gamma_N^{\text{uni}}$ such that $0 \in \partial P''$, $\mathbb{R}^N \setminus P''$ is convex, $z \in \partial P'$, P' is convex, $P' \subseteq P$, and $P = P' + P''$. Indeed, as 0 is not in the interior of P'' , $\{0\} = \phi(P'', 0) = \sigma(P'', 0)$. Moreover, $z \in \phi(P', r) = \sigma(P', r)$ by IIE of ϕ and Case (5) so that $y \in \sigma(P, r)$ by CADD. Now, we shall construct P' and P'' . Let $G^{P'}, G^{P''} : \mathbb{R}^N \rightarrow \mathbb{R}^N$ be defined by

$$G^{P'}(x) = 2y - x, G^{P''}(x) = 2x - 2y \quad \forall x \in \mathbb{R}^N$$

and define

$$P' = \{z \in \mathbb{R}^N \mid \exists x \in \partial P : z \leq G^{P'}(x)\}, P'' = \{z \in \mathbb{R}^N \mid \exists x \in \partial P : z \leq G^{P''}(x)\}.$$

With $\lambda = \lambda^{y, P}$ and $g = g^{\lambda, P}$ it may easily be deduced that, for all $x \in \mathbb{R}^N$,

$$g^{\lambda, P'}(x) = -g(2y - x) \text{ and } g^{\lambda, P''}(x) = 2g\left(\frac{x}{2} + y\right)$$

so that $\nabla g^{\lambda, P'}(x) = \nabla g(2y - x)$ and $\nabla g^{\lambda, P''}(x) = \nabla g\left(\frac{x}{2} + y\right)$. Hence, P' and P'' are uniformly p -smooth by Remark A.1 of the appendix. It is straightforward to verify that P' and P'' satisfy the desired properties.

- (7) Now we are able to consider the general case and prove that $\sigma(P, r) = \phi(P, r)$.

- (a) Let $y \in \sigma(P, r)$. By EFF and Corollary A.4 there exists $(\tilde{P}, r) \in \Gamma_N^{\text{uni}}$ such that \tilde{P} is convex, $y \in \partial \tilde{P}$, and $\tilde{P} \subseteq P$. By IIA, $y \in \sigma(\tilde{P}, r)$. By Case (5), $y \in \phi(\tilde{P}, r)$. By IIE of ϕ , $y \in \phi(P, r)$.
- (b) In order to show the opposite inclusion, let $z \in \phi(P, r)$. By EFF, $z \in \partial P$. By Corollary A.4 there exists $(P', -r) \in \Gamma_N^{\text{uni}}$ such that P' is convex, $P' \subseteq -(\mathbb{R}^N \setminus P)$, and $-z \in P'$. Let $P'' = -(\mathbb{R}^N \setminus P')$. By IIE of ϕ , $z \in \phi(P'', r)$. By Case (6), $z \in \sigma(P'', r)$. By IIA, $z \in \sigma(P, r)$. **q.e.d.**

4 On the Logical Independence of the Axioms

The following solution, denoted by $\widehat{\phi}$, will be useful. For $(P, r) \in \Gamma_N^{\text{uni}}$ let

$$\widehat{\phi}(P, r) = \begin{cases} \phi(P, r) & , \text{ if } r \in P, \\ \arg \max \{ \prod_{i \in N} (r_i - x_i) \mid x \in \partial P, x \leq r \} & , \text{ if } r \notin P. \end{cases}$$

By Proposition 2.4 and a careful inspection of Claim (2) of its proof, $\widehat{\phi}$ satisfies NE. In view of and EFF and UNA, because $\widehat{\phi}$ is a nonempty subsolution of ϕ . It is straightforward to verify that $\widehat{\phi}$ satisfies SCOV, TCOV, and IIE. The following example shows that $\widehat{\phi}$ does not satisfy IIA provided that $|N| \geq 2$.

Example 4.1 Let $|N| \geq 2$ and $X = \{x \in \mathbb{R}^N \mid x \leq 0, \prod_{i \in N} (-x_i) \geq 1\}$. Then X is a p-smooth feasible set. If $Y = \{y \in \partial X \mid y_i \geq -2\}$, then $Y \neq \emptyset$ so that $P = \{z \in \mathbb{R}^N \mid \lambda^{y, X} \cdot z \leq \lambda^{y, X} \cdot y \ \forall y \in Y\}$ is uniformly p-smooth. Let $r = 0$. We may easily deduce that $\phi(P, r) = Y$. Let $P' = P - \{\chi^N\}$. By symmetry of P , $\phi(P', r) \ni -2\chi^N$. Let $i \in N$. Define \widehat{x} by $\widehat{x}_j = -3$ and $\widehat{x}_i = -1 - \frac{1}{2^{|N|-1}}$ and observe that $\widehat{x} \in \partial P'$. However, $\prod_{i \in N} -\widehat{x}_i = 3^{n-1} + \frac{3^{n-1}}{2^{n-1}} > 2^n = \prod_{i \in N} 2$ so that $-2\chi^N \notin \widehat{\phi}(P', r)$. For sketches of P and P' in the case $|N| = 2$ see Figure 4.1.

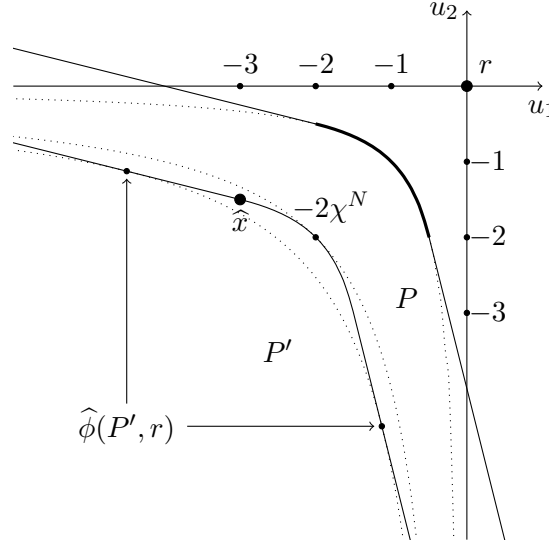


Figure 4.1: Sketch of the uniformly p-smooth feasible sets P and P'

We now show that $\widehat{\phi}$ satisfies CADD.

Lemma 4.2 *The solution $\widehat{\phi}$ on Γ_N^{uni} satisfies CADD.*

Proof: For $i \in \{1, 2\}$, let $(P^i, r^i) \in \Gamma_N^{\text{uni}}$, $x^i \in \widehat{\phi}(P^i, r^i)$ such that, with $P = P^1 + P^2$, $r = r^1 + r^2$, and $x = x^1 + x^2$, $(P, r) \in \Gamma_N^{\text{uni}}$ and $x \in \partial P$. By CADD of ϕ , $x \in \phi(P, r)$. It remains to show that $x \in \widehat{\phi}(P, r)$. If $r \in P$, then the proof is finished.

Hence, we may assume that $r \notin P$. As $x \in \partial P$, $\lambda^{x^1, P^1} = \lambda^{x^2, P^2} = \lambda^{x, P}$. By Remark 2.3, there exists $c \in \mathbb{R}$ such that $(r^2 - x^2) = c(r^1 - x^1)$. A careful inspection of the proof of Proposition 2.6 (see Claims (1) through (3)) shows that $x^1 \ll r^1$ or $x^2 \ll r^2$. Without loss of generality we may assume that $x^1 \ll r^1$. By definition of $\widehat{\phi}$,

$$P^1 \supseteq \left\{ z \in \mathbb{R}^N \mid z \ll r^1, \prod_{i \in N} (r_i^1 - z_i) \geq \prod_{i \in N} (r_i^1 - x_i^1) \right\} =: Z^1 \quad (4.1)$$

Let $Z = \{z \in \mathbb{R}^N \mid z \ll r, \prod_{i \in N} (r_i - z_i) \geq \prod_{i \in N} (r_i - x_i)\}$. Two cases may occur:

- (1) $x^2 \geq r^2$. By Expression (4.1), $P \supseteq \{x^2\} + Z^1$. Let $z \in Z$ and define $z^1 = z - x^2$. It suffices to show that $z^1 \in Z^1$. Now, $z^1 \ll r^1$, because $x^2 \geq r^2$ and $z \ll r$. As $r \gg x$, $-1 < c \leq 0$. With $a = r - z, b = r - x$, and $\alpha = -\frac{c}{1+c}$, we receive

$$r^1 - z^1 = a + \alpha b, r^1 - x^1 = (1 + \alpha)b, \quad a, b > 0, \quad \text{and } \alpha \geq 0.$$

Hence, it suffices to show the following implication:

$$a, b \in \mathbb{R}_{++}^N, \prod_{i \in N} a_i \geq \prod_{i \in N} b_i, \alpha \geq 0 \implies \prod_{i \in N} (a_i + \alpha b_i) \geq \prod_{i \in N} ((1 + \alpha)b_i). \quad (4.2)$$

Indeed, Implication (4.2) may be proved by induction on $|N|$. If $|N| = 1$, then the desired implication is obviously valid. Assume that Implication (4.2) is valid for $|N| \leq r$ for some $r \in \mathbb{N}$. If $|N| = r + 1$, then we may assume that $a_k < b_k$ for some $k \in N$ and, hence, $a_\ell > b_\ell$ for some $\ell \in N$. Define $\tilde{a} \in \mathbb{R}^N$ by $\tilde{a}_i = a_i$ for all $i \in N \setminus \{k, \ell\}$, $\tilde{a}_k \tilde{a}_\ell = a_k a_\ell$, and

$$\tilde{a}_k = \begin{cases} b_k & , \text{ if } a_k a_\ell \leq b_k b_\ell, \\ \frac{a_k a_\ell}{b_\ell} & , \text{ otherwise.} \end{cases}$$

Then

$$\tilde{a} \gg 0, \prod_{i \in N} \tilde{a}_i = \prod_{i \in N} a_i, \text{ and } \prod_{i \in N} (\tilde{a}_i + \alpha b_i) \leq \prod_{i \in N} (a_i + \alpha b_i).$$

The proof is finished by applying the inductive hypothesis, either to $N \setminus \{k\}$, if $\tilde{a}_k = b_k$, or to $N \setminus \{\ell\}$ in the other case, i.e., if $\tilde{a}_\ell = b_\ell$.

- (2) $x^2 \ll r^2$. Let

$$Z^2 = \left\{ z \in \mathbb{R}^N \mid z \ll r^2, \prod_{i \in N} (r_i^2 - z_i) \geq \prod_{i \in N} (r_i^2 - x_i^2) \right\}$$

By definition of $\widehat{\phi}$, $P^2 \supseteq Z^2$. As $Z^1 + Z^2 \supseteq Z$, the proof is finished.

q.e.d.

The solution $\sigma^i, i = 1, \dots, 6$, shows that the i -th axiom in Theorem 3.1 is logically independent of the remaining axioms provided that $|N| \geq 2$. If $(P, r) \in \Gamma_N^{\text{uni}}$, then

$$\begin{aligned}\sigma^1(P, r) &= \begin{cases} \phi(P, r) & , \text{ if } r \in P \setminus \partial P, \\ \emptyset & , \text{ otherwise;} \end{cases} \\ \sigma^2(P, r) &= \begin{cases} \phi(P, r) & , \text{ if } r \in P \setminus \partial P, \\ \phi(P, r) - \mathbb{R}_+^N & , \text{ otherwise;} \end{cases} \\ \sigma^3(P, r) &= \begin{cases} \phi(P, r) & , \text{ if } r \in P \setminus \partial P, \\ \partial P & , \text{ otherwise;} \end{cases} \\ \sigma^4(P, r) &= \partial P; \\ \sigma^5(P, r) &= \{x \in \partial P \mid x = r + t\chi^N \text{ for some } t \in \mathbb{R}\}; \\ \sigma^6(P, r) &= \phi(P, 0).\end{aligned}$$

As mentioned in Corollary 2.6, the Nash solution satisfies IIE. Hence the seven axioms employed in Theorem 3.1 imply IIE. For completeness reasons we should like to remark that all of the axioms are needed to guarantee IIE. Indeed, each of the foregoing examples $\sigma^1, \sigma^2, \sigma^4, \sigma^5, \sigma^6$, and $\widehat{\phi}$ satisfies IIE. In order to show that also CADD is needed, we may replace σ^3 by the solution σ' that differs from σ^3 only inasmuch as it assigns $\{x \in \partial P \mid x \leq r\}$ to each (P, r) with $r \notin P \setminus \partial P$. The solution σ' inherits NE, EFF, UNA, SCOV, TCOV, and IIA from σ^3 , and it satisfies IIE in addition.

5 Bargaining problems and p-smooth choice problems

This section is devoted to some modifications of Theorem 3.1 showing the robustness of this axiomatization.

We first treat p-smooth choice problems with convex feasible sets. Let $\Gamma_N^{\text{con}} = \{(P, r) \in \Gamma_N \mid P \text{ is convex}\}$. A set $\Gamma \subseteq \Gamma_N^{\text{con}}$ is a *feasible domain* of convex choice problems if

- (1) $\Gamma \subseteq \Gamma_N^\phi$;
- (2) $(\{x \in \mathbb{R}^N \mid x(N) \leq c\}, r) \in \Gamma$ for all $c \in \mathbb{R}$ and all $r \in \mathbb{R}^N$ such that $r(N) \leq c$;
- (3) If $(P, r) \in \Gamma$ and $\lambda \in \mathbb{R}_{++}^N$, then $(\lambda * P, \lambda * r) \in \Gamma$;
- (4) If $(P, r) \in \Gamma$ and $x \in \partial P$, then $(\{y \in \mathbb{R}^N \mid \lambda^{x, P} \cdot y \leq \lambda^{x, P} \cdot x\}) \in \Gamma$.

A careful inspection of the cases (1) through (5) in the proof of Theorem 3.1 shows that this result holds when replacing Γ_N^{uni} by any feasible domain of convex choice problems, i.e., we have the following proposition.

Proposition 5.1 *Let $\Gamma \subseteq \Gamma_N^{\text{con}}$ be a feasible domain. The Nash solution on Γ is the unique solution that satisfies NE, EFF, CADD, UNA, SCOV, TCOV, and IIA.*

A choice problem with reference point (P, r) is a *bargaining problem* if $r \in P$. In this case, we may interpret r as the *disagreement point* and use the letter d instead of r . A bargaining problem (P, d) is proper if d is an element of the interior of P .

A careful inspection of the proof of Theorem 3.1 shows that this result still holds if Γ_N^{uni} is replaced by the subset of uniformly p-smooth bargaining problems on N .

Moreover, it should be noted that the set of bargaining problems $(P, d) \subseteq \Gamma_N$ such that P is convex is a feasible domain so that Proposition 5.1 applies also to this set. However, in this case, we don't need IIA.

Theorem 5.2 *Let $\Gamma \subseteq \Gamma_N^{\text{con}}$ consist of bargaining problems only. Then $\Gamma \subseteq \Gamma_N^\phi$ and if Γ is feasible, then the Nash solution on Γ is the unique solution that satisfies NE, EFF, CADD, UNA, SCOV, and TCOV.*

Proof: Let $(P, d) \in \Gamma$. By (1) of Remark 2.7, $\phi(P, d) \neq \emptyset$. In order to show the second statement, we may follow the first part of the proof of Theorem 3.1. Indeed, we may first literally the proof until and including case (3) because IIA is not employed in this part. Note that cases 4, 6, and 7 cannot occur. The first inclusion $\sigma(P, d) \subseteq \phi(P, d)$ of case (5) may be literally copied—it just refers to IIA of ϕ . The part that shows the other inclusion with the help of IIA may now be circumvented, because by (1) of Remark 2.7, $\phi(P, d)$ is a singleton. **q.e.d.**

We now check if our result may be applied to Γ_N^ϕ . If we want to adjust the proof of Theorem 3.1 suitably, we first notice that we have to replace Corollary A.4 by Lemma A.3 wherever it occurs. Unfortunately, the general case (7) may not be adjusted in a straightforward manner. In fact, if P is not uniformly p-smooth, then the convex p-smooth choice problem (\tilde{P}, r) whose existence is guaranteed by Lemma A.3 may not be uniformly p-smooth so that $\phi(\tilde{P}, r) = \emptyset$ might be possible.

In order to overcome this problem we employ a slightly stronger version of IIA. For the sake of completeness, we also modify IIE. Let $\Gamma \subseteq \Gamma_N$ and σ be a solution on Γ . Then σ satisfies

- (7') IIA' if $(P, r) \in \Gamma_N$, $(P', r) \in \Gamma$, $P \subseteq P'$, and $\sigma(P', r) \cap P \neq \emptyset$ imply that $(P, r) \in \Gamma$ and $\sigma(P', r) \cap P \subseteq \sigma(P, r)$;
- (8') IIE' if $(P', r) \in \Gamma_N$, $(P, r) \in \Gamma$, $P \subseteq P'$, and $\sigma(P, r) \cap \partial P' \neq \emptyset$ imply that $(P', r) \in \Gamma$ and $\sigma(P, r) \cap \partial P' \subseteq \sigma(P', r)$.

Note that ϕ satisfies IIA' and IIE' on Γ_N^ϕ . Moreover, if we replace IIA and IIE by the stronger versions wherever they occur in the proof of Theorem 3.1, then we receive the following result.

Corollary 5.3 *On the set Γ_N^ϕ the Nash solution is the unique solution that satisfies NE, EFF, CADD, UNA, SCOV, TCOV, and IIA’.*

It should be remarked that Corollary 5.3 still holds when Γ_N^ϕ is replaced by the subset of all bargaining problems in Γ_N^ϕ .

Note that the solutions σ^i , $i = 1, \dots, 5$, defined in Section 4 still show that the i -th axiom is logically independent of the remaining axioms in all foregoing characterizations of the Nash solution provided $|N| \geq 2$. Moreover, σ^6 still violates TCOV and satisfies EFF, CADD, UNA, SCOV, and IIA’ on all mentioned classes of choice problems. However, σ^6 also violates NE if Γ contains all bargaining problems in Γ_N^ϕ so that, e.g., we don’t know if TCOV is really needed in Corollary 5.3.

We remark that the logical independence of IIA’ in Corollary 5.3 is an open problem.

Finally, it should be noted that our solution $\hat{\phi}$ defined in Section 4 still shows that IIA is needed in Proposition 5.1 if the feasible domain consists, e.g., of all convex *uniformly* p -smooth choice problems. However, in general it is an open problem if IIA is really needed. Only in the case $|N| = 2$ it is known that IIA is redundant if $\Gamma = \Gamma_N^{\text{con}} \cap \Gamma_N^\phi$ (Peleg, Sudhölter, and Zarzuelo 2012, Theorem 3.1).

A Appendix

The following remarks are useful.

Remark A.1 Let $N \neq \emptyset$ be finite and $\lambda \in \mathbb{R}_{++}^N$.

- (1) Let $P \subseteq \mathbb{R}^N$ be feasible. Then there is a unique function $g = g^{\lambda, P} : \mathbb{R}^N \rightarrow \mathbb{R}$ that satisfies $x - g(x)\lambda \in \partial P$ for all $x \in \mathbb{R}^N$. The mapping g is continuous and satisfies

$$P = \{x \in \mathbb{R}^N \mid g(x) \leq 0\}; \quad (1)$$

$$g(x + t\lambda) = g(x) + t \quad \forall x \in \mathbb{R}^N \quad \forall t \in \mathbb{R}; \quad (2)$$

$$g \text{ is nondecreasing}; \quad (3)$$

$$P \text{ is smooth} \Rightarrow \nabla g(x) = \frac{\lambda^{x-g(x)\lambda, P}}{\lambda \cdot \lambda^{x-g(x)\lambda, P}} \quad \forall x \in \mathbb{R}^N, \quad (4)$$

where, for any differentiable real function $f : \mathbb{R}^N \rightarrow \mathbb{R}$, $\nabla f = \left(\frac{\partial f}{\partial x_i} \right)_{i \in N}$ denotes the gradient⁴ of f .

⁴By (2) of Definition 2.1, the function $g^{\lambda, P}$ is C^1 provided that P is smooth.

- (2) Let $g : \mathbb{R}^N \rightarrow \mathbb{R}$ be a continuous mapping that satisfies Eq. (2) and Condition (3). Then the set $P^g = P$ defined by Eq. (1) is feasible and $g^{\lambda, P} = g$. Moreover, if ∇g is continuous, then P is smooth, and g is convex if and only if P is convex.

Remark A.2 Let $P, Q \in \mathbb{R}^N$ be convex such that P, Q , and $P + Q$ are feasible.

- (1) If P is smooth, then $P + Q$ is also smooth.
- (2) If P is smooth and, for any $y \in \partial Q$ and any $\lambda \in \mathbb{R}^N$ such that $\lambda > 0$ and such that $H = \{y' \in \mathbb{R}^N \mid \lambda \cdot y' = \lambda \cdot y\}$ is a supporting hyperplane at y of Q , $\lambda \gg 0$, then $P + Q$ is p-smooth.
- (3) If P is smooth and there exists $\varepsilon > 0$ such that, for any $y \in \partial Q$ and any $\lambda \in \mathbb{R}^N$ such that $\lambda > 0$ and $\lambda(N) = 1$ and such that $H = \{y' \in \mathbb{R}^N \mid \lambda \cdot y' = \lambda \cdot y\}$ is a supporting hyperplane at y of Q , $\lambda \gg \varepsilon \chi^N$, then $P + Q$ is uniformly p-smooth.

Indeed, in order to verify Remark A.2, note that any $z \in \partial(P + Q)$ is of the form $z = x + y$ for suitable $x \in \partial P$ and $y \in \partial Q$ and if $H = \{z' \in \mathbb{R}^N \mid \lambda \cdot z' = \lambda \cdot z\}$ for some $\lambda \geq 0, \lambda \neq 0$, is a supporting hyperplane at z to $P + Q$, then $\{x' \in \mathbb{R}^N \mid \lambda \cdot x' = \lambda \cdot x\}$ is a supporting hyperplane of P at x so that λ is proportional to $\lambda^{x, P}$ and $\{y' \in \mathbb{R}^N \mid \lambda \cdot y' = \lambda \cdot y\}$ is a supporting hyperplane at y to Q . These observations immediately imply the desired statements.

Lemma A.3 *If $P \subseteq \mathbb{R}^N$ is a p-smooth feasible set and $y \in \partial P$, then there exists a p-smooth feasible set $\tilde{P} \subseteq \mathbb{R}^N$ such that*

$$\tilde{P} \text{ is convex}; \quad (5)$$

$$y \in \partial \tilde{P}; \quad (6)$$

$$\tilde{P} \subseteq P. \quad (7)$$

Proof: For simplicity we may assume that $y = 0$ (otherwise replace P by $P - \{y\}$ and y by 0). Moreover, we may assume that $\lambda^y = \lambda^{y, P} = \frac{1}{n} \chi^N$ (otherwise replace P by $\lambda^y * P$). Let $g = g^{\lambda^y, P}$ and $X = H^{y, P} = \{x \in \mathbb{R}^N \mid \lambda^y \cdot x = 0\} = \{x \in \mathbb{R}^N \mid x(N) = 0\}$. Now we define $G : \mathbb{R}_+ \rightarrow \mathbb{R}$ by the requirement that

$$G(t) = \max\{\nabla g(rz) \cdot z \mid z \in X, \|z\| = 1, 0 \leq r \leq t\} \quad \forall t \geq 0, \quad (8)$$

and note that G is well-defined by Remark A.1. Moreover, $G(0) = 0$ and G is continuous. By the fundamental theorem of calculus there exists a function $\hat{g} : \mathbb{R} \rightarrow \mathbb{R}$ such that $\hat{g}' = G$ and $\hat{g}(0) = 0$. As G is monotonically increasing, \hat{g} is a convex function. For any $z \in X$ satisfying $\|z\| = 1$, define $g_z : \mathbb{R} \rightarrow \mathbb{R}$ by $g_z(t) = g(tz)$ for all $t \in \mathbb{R}_+$. Then, for all $t \geq 0$,

$$g'_z(t) = \nabla g(tz) \cdot z \leq \hat{g}'(t) \text{ and } g_z(0) = 0.$$

Hence, we may conclude that

$$g_z(t) \leq \widehat{g}(t) \quad \forall t \in \mathbb{R}_+. \quad (9)$$

Now we are ready to define \bar{g} as follows: For $z \in \mathbb{R}^N$ let $x(z) = z - z(N)\lambda^y$. Then $x(z) \in X$ and we define $\bar{g}(z) = \widehat{g}(\|x(z)\|) + z(N)$ so that $\bar{g}(z + t\lambda^y) = \bar{g}(z) + t$ for $t \in \mathbb{R}$. By Eq. (8), \bar{g} is symmetric in the sense that

$$\bar{g}(z) = \bar{g}\left(z + \chi^{\{i\}}(z_j - z_i) + \chi^{\{j\}}(z_i - z_j)\right) \text{ for all } z \in \mathbb{R}^N \text{ and } i, j \in N. \quad (10)$$

By Inequality (9) and Eq. (2),

$$g(z) \leq \bar{g}(z) \quad \forall z \in \mathbb{R}^N. \quad (11)$$

Define $\widehat{P} = \{z \in \mathbb{R}^N \mid \bar{g}(z) \leq 0\} - \mathbb{R}_+^N$. By Inequality (11) and comprehensiveness of P , $\widehat{P} \subseteq P$. Moreover, $y = 0 \in \widehat{P}$. It is easy to check that, for all $z \in \mathbb{R}^N$ and all $i \in N$,

$$\frac{\partial \bar{g}}{\partial x_i}(z) = \begin{cases} 1 & , \text{ if } x(z) = 0, \\ \widehat{g}'(\|x(z)\|) \frac{x_i(z)}{\|x(z)\|} + 1 & , \text{ if } x(z) \neq 0. \end{cases} \quad (12)$$

As \widehat{g} is a convex function, we conclude that \widehat{P} is closed and convex. In order to show that \widehat{P} is smooth, by convexity, it suffices to show that at each $z \in \partial \widehat{P}$ there exists a unique supporting hyperplane to \widehat{P} . Let $\lambda' \in \mathbb{R}^N$, $\lambda' \neq 0$, be some vector such that $\widehat{P} \subseteq \{z' \in \mathbb{R}^N \mid \lambda' \cdot z' \leq \lambda' \cdot z\}$. Then there exist $\widehat{x} \in \mathbb{R}^N$ with $\bar{g}(\widehat{x}) \leq 0$ and $x \in -\mathbb{R}_+^N$ such that $z = \widehat{x} + x$. Hence $\bar{g}(\widehat{x}) = 0$ and $x \in \partial(-\mathbb{R}_+^N)$. As $-\mathbb{R}_+^N$ is a cone, $\lambda' \cdot x \leq 0$. As $\lambda' \cdot \widehat{x} = \lambda' \cdot (\widehat{x} + 0x) \leq \lambda' \cdot z = \lambda' \cdot \widehat{x} + \lambda' \cdot x$, we may conclude that $\lambda' \cdot x = 0$ so that $\lambda' \cdot \widehat{x} \geq \lambda' \cdot z'$ for all $z' \in \mathbb{R}^N$ with $\bar{g}(z') = 0$. Thus, λ' is proportional to $\nabla \bar{g}(z)$.

Now, \widehat{P} is p-smooth, then $\widetilde{P} = \widehat{P}$ satisfies the desired properties. Hence, we may assume that \widehat{P} is not p-smooth. We now construct a p-smooth feasible set \widetilde{P} with $\frac{1}{2}\widehat{P} \subseteq \widetilde{P} \subseteq \widehat{P}$. Note that $\frac{1}{2}\widehat{P}$ is a smooth convex feasible set that contains $y = 0$ in its boundary. Moreover, $g^{\lambda^y, \frac{1}{2}\widehat{P}}(z) = \frac{\bar{g}(2z)}{2}$ for all $z \in \mathbb{R}^N$. For $i \in N$, let $X^i = \{x \in X \mid \frac{\partial \bar{g}}{\partial z_i}(x) \leq \frac{1}{2}\}$. As \widehat{P} is not p-smooth, by Remark A.1 (4), a careful inspection of Eq. (12) shows that $X^i \neq \emptyset$ for all $i \in N$. Let $x, x' \in X^i$ such that $\|x'\| \geq \|x\|$. As \widehat{g} is convex and $\bar{g}(z) = \widehat{g}(\|x(z)\|)$ for all $z \in X$, $\bar{g}(x')/2 - \bar{g}(x'/2) \geq \bar{g}(x)/2 - \bar{g}(x/2)$. As X^i is closed, we may conclude that

$$\alpha_i = \min_{x \in X^i} \frac{\bar{g}(x)}{2} - \bar{g}\left(\frac{x}{2}\right)$$

exists. Moreover, by smoothness of \widehat{P} , $\alpha^i > 0$. Finally, by Eq. (10) we may conclude that $\alpha_i = \alpha_j$ for all $i, j \in N$. Now we are ready to construct \widetilde{P} . Let $\varepsilon \leq \min \left\{ \frac{\alpha_i}{|N|}, \frac{1}{2|N|} \right\}$ for all $i \in N$ and define

$$Q = \left\{ y \in \mathbb{R}^N \mid y^+(N) \leq \varepsilon \frac{y^-(N)}{1 + y^-(N)} \right\},$$

where x^+ and x^- denote the positive and negative part of any $x \in \mathbb{R}^N$, i.e., $x^+, x^- \in \mathbb{R}_+^N$ such that $x = x^+ - x^-$. Note that Q is convex and feasible and that $Q \subseteq (-\mathbb{R}_+^N) + \{\varepsilon \chi^N\}$. Let

$\tilde{P} = \frac{1}{2}\hat{P} + Q$. It is straightforward to verify that \tilde{P} is closed so that \tilde{P} is feasible. We conclude that Q satisfies the assumptions in (2) of Remark A.2. It suffices to show that $\tilde{P} \subseteq \hat{P}$. Let $z \in \partial\tilde{P}$. Then there exist $x \in \partial\left(\frac{1}{2}\hat{P}\right)$, $y \in \partial Q$, and $\lambda \in \mathbb{R}_+^N$ with $\lambda(N) = 1$ so that $z = x + y$ and $\frac{1}{2}\hat{P} \subseteq \{x' \in \mathbb{R}^N \mid x' \cdot \lambda \leq x \cdot \lambda\}$ and $Q \subseteq \{y' \in \mathbb{R}^N \mid y' \cdot \lambda \leq y \cdot \lambda\}$. If $|N| \min_{i \in N} \lambda_i > \frac{1}{2}$, then

$$\begin{aligned} \lambda \cdot y &\leq \frac{-y^-(N)}{2|N|} + y^+(N) && \leq \frac{-y^-(N)}{2|N|} + \varepsilon \frac{y^-(N)}{1+y^-(N)} \\ &\leq y^-(N) \left(\frac{\varepsilon}{1+y^-(N)} - \frac{2}{2|N|} \right) && \leq y^-(N) \left(\varepsilon - \frac{1}{2|N|} \right) \leq 0 = \lambda \cdot 0 \end{aligned}$$

so that $y = 0$ and, hence, $z \in \hat{P}$. If $|N| \min_{i \in N} \lambda_i \leq \frac{1}{2}$, then, by construction, $x + \varepsilon \chi^N \in \hat{P}$. As $y \leq \varepsilon \chi^N$, $x + y \in \hat{P}$. **q.e.d.**

Corollary A.4 *Let $P \subseteq \mathbb{R}^N$ be a uniformly p-smooth feasible set and $y \in \partial P$. Then there exists a uniformly p-smooth feasible set $\tilde{P} \subseteq \mathbb{R}^N$ that satisfies Conditions (5) through (7).*

Proof: By Lemma 2.1 there exists a p-smooth feasible set \tilde{P} with the desired properties.

We shall now modify \tilde{P} as follows: For $\mu > 0$ let

$$X_\mu = \left\{ z \in \mathbb{R}^N \mid \left(\mu \chi^{N \setminus \{i\}} + (1 - (|N| - 1)\mu) \chi^{\{i\}} \right) \cdot z \leq 0 \ \forall i \in N \right\}.$$

As P is uniformly p-smooth, for any $\mu > 0$ with $\mu < \min\{\lambda_i^y \mid i \in N\}$,

$$P + X_\mu \subseteq P. \tag{13}$$

Define $\hat{P} = \tilde{P} + X_\mu$. As a sum of two convex sets, \hat{P} is convex. By Expressions (13) and (11), $\hat{P} \subseteq P$, and $y \in \hat{P}$, because $y \in \tilde{P}$. As X_μ is a convex cone, $\{x\} + X_\mu \subseteq \hat{P}$ for all $x \in \tilde{P}$. As $X_\mu \supseteq -\mathbb{R}_+^N$, \hat{P} is comprehensive.

We now show that \hat{P} is closed. Let $(\hat{x}^t + x^t)_{t \in \mathbb{N}}$ be a convergent \mathbb{R}^N -sequence such that $\hat{x}^t \in \tilde{P}$ and $x^t \in X_\mu$ for all $t \in \mathbb{N}$. It remains to prove that $z = \lim_{t \rightarrow \infty} \hat{x}^t + x^t \in \hat{P}$. If $z' \in \mathbb{R}^N$ such that $\hat{x}^t + x^t \geq z'$ for all $t \in \mathbb{N}$, then $\hat{x}^t \in \{z'\} - X_\mu$ for all $t \in \mathbb{N}$. As $\mu < \lambda_i^y$ for all $i \in N$, $\{x \in \mathbb{R}^N \mid \lambda^y \cdot x \leq \lambda^y \cdot y\} \cap (\{z'\} - X_\mu)$ is compact. As $\tilde{P} \subseteq \{x \in \mathbb{R}^N \mid \lambda^y \cdot x \leq \lambda^y \cdot y\}$, $\tilde{P} \cap (\{z'\} - X_\mu)$ is also compact. Hence, $(\hat{x}^t)_{t \in \mathbb{N}}$ has a convergent subsequence, let us say, is convergent itself. We conclude that $(x^t)_{t \in \mathbb{N}}$ is also convergent. As \tilde{P} and X_μ are closed, $\lim_{t \rightarrow \infty} \hat{x}^t \in \tilde{P}$ and $\lim_{t \rightarrow \infty} x^t \in X_\mu$ so that $z = \lim_{t \rightarrow \infty} \hat{x}^t + \lim_{t \rightarrow \infty} x^t \in \hat{P}$. Hence \hat{P} is feasible and Remark A.2 finishes the proof. **q.e.d.**

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